Lesson-4: Business Applications of Differentiation

Lesson Objectives:

After studying this lesson, you should be able to:

- State the key concepts related to business applications of differentiation;
- Apply the techniques of differentiation to solve business problems.

Key Concepts

Total costs (TC): Total cost is the combination of fixed cost and variable cost of output. If the production increases, only total variable cost will increase in direct proportion but the fixed cost will remain unchanged within a relevant range.

Total revenue (TR): Total revenue is the product of price/demand function and output.

Profit: Profits are defined as the excess of total revenue over total costs. Symbolically it can be expressed as, P (profit) = TR – TC. i.e., (Total Revenue – Total Cost)

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The rules for finding a maximum point tell us that P is maximized when the derivative of the profit function is equal to zero and the second derivative is negative. If we denote the derivatives of the revenue and cost functions by dTR and dTC we have,

'P' is at a maximum when dTR - dTC = 0. This equation may be written as dTR = dTC.

The derivative of the total revenue function must be equal to the derivative of the total cost function for profits to be maximized.

Profit Maximizing output = $\frac{d(\text{profit function})}{dx}$

Condition: In case of maximization, the conditions are $\frac{dp}{dx} = 0$ and $\frac{d^2p}{dx^2}$ must be Negative.

Cost Minimizing output = $\frac{d(\text{total cost function})}{dx}$

Condition: In case of minimization, the conditions are $\frac{dTCy}{dx} = 0$ and $\frac{d^2TC}{dx^2}$ must be Positive.

Marginal Cost (MC): MC is the extra cost for producing one additional unit when the total cost at certain level of output is known. Hence, it is the rate of change in total cost with respect to the level of output at the point where total cost is known. Therefore, we have, $MC = \frac{dTC}{dx}$ where total cost (TC) is a function of x, the level of output.

Marginal Production (MP): MP is the incre–mental production, i.e., the additional production added to the total production, i.e. $MP = \frac{dTP}{dx}$

Marginal Revenue (MR): MR is defined as the change in the total revenue for the sale of an extra unit. Hence, it is the rate of change total in revenue with respect to the quantity demanded at the point where total revenue is known. Therefore, we have, $MR = \frac{dTR}{dx}$ where total revenue (TR) is a function of x, the quantity demanded.

> *MR is defined as the change in the total revenue for the sale of an extra unit.*

Let us illustrate these concepts by the following examples.

Example–1:

The profit function of a company can be represented by $P = f(x) = x - 0.00001x^2$, where x is units sold. Find the optimal sales volume and the amount of profit to be expected at that volume.

Solution:

The necessary condition for the optimal sales volume is that the first derivative of the profit function is equal to zero and the second derivative must be negative. Where the profit function is:

$P = x - 0.00001x^2$

$$\frac{dP}{dx} = \frac{d}{dx}\left(x - 0.00001x^2\right)$$

Marginal Profit = 1 – 0.00002x

To get maximum profit now we put marginal profit = 0

So, $1 - 0.00002x = 0$

or, $0.00002x = 1$

So, $x = 0.00002 = 50,000$ units.

The second derivative of profit function, i.e.

$$\frac{d^2P}{dx^2} = \frac{d}{dx}\left(1 - 0.00002x\right) = -0.00002 < 0$$

Now by putting the value of x in profit function we get maximum profit.

$P = x - 0.00001x^2$

$$= 50,000 - 0.00001(50,000)^2 = 50,000 - 0.00001(2500000000)$$

= 50,000 – 25,000 = 25,000.

The optimum output for the company will be 50,000 units of x and maximum profit at that volume will be Tk.25,000.

Example – 2:

If the total manufacturing cost 'y' (in Tk.) of making x units of a product is : y = 20x + 5000, (a) What is the variable cost per unit? (b) What is the fixed cost? (c) What is the total cost of manufacturing 4000 units? (a) What is the marginal cost of producing 2000 units?

Solution:

We have the cost-output equation : y = 20x + 5000. We know that, if the production increases, only total variable cost will increase in direct proportion but the fixed cost will remain unchanged in total. So, the derivative of y with respect to the increase in x by 1 unit will give the variable cost per unit.

(a) Variable cost per unit $= \frac{d}{dx}\left(\text{cost-output equation}\right)$

$$= \frac{d}{dx}(y)$$

$$= \frac{d}{dx}(20x + 5000) = 20.$$

∴ Variable cost per unit is Tk.20.

(b) Total fixed cost will remain unchanged even if we don't produce any unit. If we don't produce any unit, there will be no variable cost and only fixed cost will be the total cost. So, if we put x=0 in the cost-output equation, we will get the fixed cost.

∴ Fixed Cost = y = $\left[20.(0) + 5,000\right]$ = Tk.5000

(c) If we put x=4000 in the cost-output equation, we will get the total cost of producing 4,000 units.

∴ Total cost of producing 4000 units = y = 20 (4,000) + 5,000 = Tk.85,000.

(d) We know that the marginal cost of 'n'th unit = $TC_n - TC_{n-1}$

∴ Marginal cost of 2000th unit

= TC of 2000 units – TC of (2000–1) units

= [20 (2000) + 5000] – [20 (1999 Tk. + 5000] = [45,000 – 44,980] = 20.

Example–3:

The total cost of producing x articles is $\frac{5}{4}x^2 + 175x + 125$ and the price at which each article can be sold is $250 - \frac{5}{4}x$. What should be the output for a maximum profit. Calculate the profit.

Solution:

Total revenue $(TR) = \left(250 - \frac{5}{4}x\right)x = 250 - \frac{5x^2}{2}$

Total cost $(TC) = \frac{5x^2}{4} + 175x + 125$

So, profit $(P) = TR - TC$

$= 250x - \frac{5x^2}{2} - \frac{5x^2}{4} - 175x - 125 = \frac{-10x^2 - 5x^2}{4} + 75x - 125 = \frac{-15x^2}{4} + 75x - 125$

$\frac{d(p)}{dx} = \frac{-30x}{4} + 75$

The necessary condition for optimization is that the first derivative of a profit function is equal to zero and second derivative must be negative. According to the assumption, we have

$\frac{-30x}{4} + 75 = 0$

or, $\frac{-30x + 300}{4} = 0$

or, $-30x + 300 = 0$

or, $-30x = -300$

or, $x = 10$

and $\frac{d^2p}{dx^2} = \frac{d}{dx}\cdot\frac{dp}{dx} = \frac{d}{dx}\left(\frac{-30x}{4} + 75\right) = \frac{-30}{4} < 0$

Therefore the profit is maximum when the output (x) is 10

Profit function $= \frac{-15x^2}{4} + 75(x) - 125$

Putting the value of x in profit function, we get,

Profit $= \frac{-15(10)^2}{4} + 75(10) - 125$

$= \frac{-1500}{4} + 750 - 125 = -375 + 750 - 125 = 750 - 500 = 250$

Hence the profit is Tk. 250.

Example–4:

The total cost function of a firm is $C = \frac{1}{3}x^3 - 5x^2 + 28x + 10$, where C is total cost and x is output. A tax at the rate of Tk.2 per unit of output is imposed and the producer adds it to his cost. If the market demand function is given by $P = 2530 - 5x$, where P is the price per unit of output, find the profit maximizing output and price.

Solution:

Total Revenue $(TR) = (2530 - 5x)\,x = 2530x - 5x^2$

Total cost $(TC) = \left(\frac{1}{3}x^3 - 5x^2 + 28x + 10\right) + (\text{Taxes i.e.} 2x)$

$$= \frac{1}{3}x^3 - 5x^2 + 28x + 10 + 2x = \frac{1}{3}x^3 - 5x^2 + 30x + 10$$

So, profit $(P) = TR - TC$

$$= 2530x - 5x^2 - \frac{1}{3}x^3 + 5x^2 - 30x - 10 = 2500x - \frac{1}{3}x^3 - 10.$$

So, $\frac{d(P)}{dx} = 2500 - \frac{3x^2}{3} = 2500 - x^2$

The necessary condition for maximization is that the first derivative of a profit function is equal to zero and second derivative must be negative.

According to the condition, we can write

$2500 - x^2 = 0$

or, $-x^2 = -2500$

So, $x = 50$

and $\frac{d^2p}{dx^2} = -2x = -2x\,50 = -100 < 0$

Hence the profit maximizing output of the firm is 50 units.

At this level, the price is given by

Price $= 2530 - 5x = 2530 - 5 \times 50 = 2530 - 250 =$ Tk.2280.

Example–5:

A motorist has to pay an annual road tax of $50 and $110 for insurance. His car does 30 miles to the gallon which costs 75 Pence (per gallon). The car is serviced every 3000 miles at a cost of $20, and depreciation is calculated in pence by multiplying the square of the mileage by 0.001.

Obtain an expression for the total annual cost. Hence find an expression for the average total cost per mile and calculate the annual mileage which will minimize the average cost per mile.

Solution:

Suppose he covers x miles in a year.

Tax per annum = \$50; Insurance per annum \$ 110

Cost of petrol = $\frac{.75x}{30}$; Service charges = $\frac{20x}{3000}$ .20

Depreciation = $\frac{0.001x^2}{100}$ (0.001 is in pence and is divided by 100 to get \$ amount)

Total cost: $C = 50+110 \frac{0.75x}{30} + \frac{20x}{3000} + 0.00001x^2$

Average TC per mile: $\frac{C}{x} = M = \frac{160}{x} + \frac{0.75}{30} + \frac{20}{3000} + 0.00001x$

$$\frac{dM}{dx} = \frac{d}{dx}\left(160x^{-1} + \frac{0.75}{30} + \frac{20}{3000} + 0.00001x\right)$$

$$= -160x^{-2} + 0 + 0 + 0.00001$$

$$= -\frac{160}{x^2} + 0.00001$$

The necessary condition for minimization of cost is $\frac{dM}{dx} = 0$, and $\frac{d^2M}{dx^2}$ must be positive.

According to the condition, we can write $\frac{-160}{x^2} + 0.00001 = 0$

$$\text{or, } \frac{160}{x^2} = 0.00001$$

$$\text{or, } 0.00001x^2 = 160$$

$$\text{or, } x = \sqrt{\frac{160}{0.00001}} = 16000000$$

$$\text{or, } x = 4000$$

The average cost is a minimum since $\frac{d^2M}{dx^2} = -160x^{-2} + 0.00001$

$$= 320x^{-3} + 0 = \frac{320}{x^3} > 0$$

So, the motorist can cover 4000 miles in a year to minimize the average cost per mile.

Example–6:

The yearly profits of ABC company are dependent upon the number of workers (x) and the number of units of advertising (y), according to the function

$$P(x,y) = 412x + 806y - x^2 - 4y^2 - xy - 50{,}000$$

(i) Determine the number of workers and the number of units in advertising that results in maximum profit.

(ii) Determine maximum profit.

Solution:

(i) To determine the values of x and y, we equate the partial derivatives of the profit function with zero.

$$P_x(x, y) = 412 - 2x - y = 0 \quad \ldots (1)$$

$$P_y(x, y) = 806 - x - 8y = 0 \quad \ldots (2)$$

The two equation are solved simultaneously to obtain the values of x and y.

$$412 - 2x - y = 0$$

$$1612 - 2x - 16y = 0$$

$$-1200 + 15y = 0$$

$$\text{or, } 15y = 1200$$

$$\therefore y = 80$$

Substituting the value of y in equation (1), we get

$$412 - 2x - 80 = 0$$

$$\text{or } -2x = -332$$

$$\therefore x = 166$$

and

$$P_{xx}(x, y) = -2 < 0$$

$$P_{yx}(x, y) = -8 < 0$$

(ii) The profit generated from using these values is:

$$P(166, 80) = 412(166) + 806(80) - (166)^2 - 4(80)^2 - (166)(80) - 50000$$

$= 68392 + 64640 - 27556 - 25600 - 13280 - 50{,}000$

$= 16595$

Example – 7:

The cost of construction (c) of a project depends upon the number of skilled workers (x) and unskilled workers (y). If cost is given by, $C(x,y) = 50000+9x^3-72xy+9y^2$

(i) Determine the number of skilled workers and unskilled workers that results in minimum cost.

(ii) Determine the minimum cost.

Solution:

To determine the number of skilled workers (x) and unskilled workers (y), we equate the partial derivatives of the cost function with zero.

$C_x(x, y) = 27x^2 - 72y = 0 \ldots (1)$

$C_y(x, y) = -72x + 18y = 0 \ldots (2)$

Solving the two equations simultaneously gives

$27x^2 - 72y = 0$

$-288x + 72y = 0$ [Multiplying equation (2) by 4]

By adding $27x^2 - 288x = 0$

or, $x(27x - 288) = 0$

or, $27x - 288 = 0$

or, $27x = 288$

$\therefore x = 10.66$ rounded to

Substituting $x = 11$ in equation (2) we get

$-72(11) + 18y = 0$

or, $-792 + 18y = 0$

or, $18y = 792$

$\therefore y = \frac{792}{18} = 44$

(ii) Putting the value of x and y in cost function, we get:

$C(11, 44) = 50000 + 9(11)^3 - 72(11)(44) + 9(44)^2$

$= 44{,}555$

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. A study has shown that the cost of producing sign pens of a manufacturing concern is given by, $C = 30 + 1.5x + 0.0008x^2$. What is the marginal cost at x = 1000 units? If the pens sells for Tk.5.00 each, for what values of x does marginal cost equal marginal revenue?

2. The demand function faced by a firm is $P=500 - 0.2x$ and its cost function is $C=25x+10,000$. Find the optimal output at which the profits of the firm and maximum. Also find the price it will charge.

3. The demand function of a profit maximizing monopolist is, $P - 3Q - 30 = 0$ and his cost function is, $C = 2Q^2 + 10Q$. If a tax of taka 5 per unit of quantity produced is imposed on the monopolist, calculate the maximum tax revenue obtained by the Government.

4. A company produces two products, x units of type –A and y units of type –B per month. If the revenue and cost equation for the month are given by-

   $R(x, y) = 11x + 14y$, $C(x, y) = x^2 - xy + 2y^2 + 3x + 4y + 10$

5. Total cost function is given by, $TC = 3Q^2 + 7Q + 12$, Where C=Cost of production, Q = output. Find

   (i) marginal cost

   (ii) Average cost if Q = 50

6. The total cost of production for the electronic module manufactured by ABC Electronics is

   $TC = 0.04Q^3 - 0.30Q^2 + 2Q + 1$

7. Determine MC, AC, AFC and TVC function. Find out the output level for which MC is minimum. What is the amount of MC, AC and TC at this level of output.

8. The transport authority of the city corporation areas has experimented with the fare structure for the city's public bus system. The new system is fixed fare system in which a passenger may travel between two points in the city for the same fare. From the survey results, system analysis have determined an appropriate demand function, $P=2000-125Q$, where Q equals to the average number of riders per hours and p equals the fare in taka.

   Required:

(i) Determine the fare which should be charged in order to maximize hourly bus for revenue.

(ii) How many rider are expected per hour under this fare?

(iii) What is the expected maximum annual revenue?